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Interpretation of Yang–Mills instantons in terms of locally conformal geometry

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Abstract
BPST instanton and multi-instanton solutions to Yang–Mills equations are studied in the spin–charge separation formalism and in the related conformal gravity formalism. In the former instantons give non-trivial solutions to a gauged Grassmannian model, in the latter the instanton solution describes a locally conformally flat doubly wrapped cigar manifold or a more complicated manifold for multi-instanton solutions.

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1. Motivation

In a recent paper, Faddeev and Niemi [1] introduced new variables for the four-dimensional $SU(2)$ Yang–Mills theory. Their change of variables can be interpreted in terms of a separation between the spin and the charge of the gauge field, similarly to the spin–charge separation used in two-dimensional theories (see, e.g., [2, 3]). Their result shows also that in the maximal Abelian gauge (MAG) the Yang–Mills theory admits a generally covariant form and relates to a gravity theory in the limit where the metric tensor is locally conformally flat. The matter content of the gravity theory is a combination of a massive vector field, an $O(3)\sigma$-model and a $G(4,2)$ Grassmannian nonlinear σ-model that describes the embedding of two-dimensional planes (2-branes) in $\mathbb{R}^4$.

Assuming that the gravity theory flows to conformally flat geometries at low energies, the Yang–Mills theory becomes the effective theory of gravity. The former is known to have a rich family of classical solutions—instantons. Hence it should be of interest to see how instantons look like from the gravity point of view, furnishing the local minima for the effective action of gravity. So, in this paper the change of variables by Faddeev and Niemi is applied to the BPST instanton and multi-instanton solutions.

It is important to realize that the above relation of the Yang–Mills theory to the gravity theory has a form, typical to dualities. It mixes the perturbative and non-perturbative sectors. In particular, the standard perturbative vacuum of the Yang–Mills theory, $A_\mu = 0$, maps to a singular geometry with divergent curvature and vanishing metric tensor, in other words, the spacetime vanishes. This does not conflict with vanishing of the action for $A_\mu = 0$ since $\sqrt{g}$ vanishes faster than R diverges. On the other hand, the flat geometry corresponds to a non-perturbative state in the Yang–Mills theory. That is why instantons, being the basic non-perturbative objects, are of interest.

In section 2, a brief review of the spin–charge separation formalism is given. In sections 3–5, the instanton and multi-instanton solutions are studied in this framework, and the focus is on a geometrical interpretation. The results are summarized in the final section 6.

2. Spin–charge separation

We start with a short description of the spin–charge decomposition for the $SU(2)$ Yang–Mills action[1]; we refer to [1] for details. We decompose the $su(2)$ vector-potential as

$$\hat{A}_\mu = A_\mu \frac{\sigma^3}{2} + X_{\mu+}\frac{\sigma^-}{2} + X_{\mu-}\frac{\sigma^+}{2}, \tag{1}$$

where we have denoted

$$\sigma^\pm = \tfrac{1}{2}(\sigma^1 \pm \mathrm{i}\sigma^2) \tag{2}$$

$$X_{\mu\pm} = A^1_\mu \pm \mathrm{i}A^2_\mu. \tag{3}$$

We fix the $SU(2)/U(1)$ gauge freedom by employing the MAG condition, locally defined as

$$\nabla^\pm_\mu X_{\mu\pm} = 0, \tag{4}$$

where the Abelian covariant derivative $\nabla^\pm$ is

$$\nabla^\pm_\mu = \partial_\mu \pm \mathrm{i}A_\mu. \tag{5}$$

In order to avoid Gribov copies, one can also demand that the gauge fields satisfy the global MAG condition

$$A : \int \left({}^{\Omega}\!A^1_\mu\right)^2 + \left({}^{\Omega}\!A^2_\mu\right)^2 \,\mathrm{d}^4x \qquad \text{is minimal at} \qquad \Omega = 1. \tag{6}$$

Here ${}^{\Omega}\!A_\mu$ is the gauge transformation of A_μ. The remaining $U(1) \in SU(2)$ gauge freedom will not be gauge fixed. Instead, we will eliminate this remaining gauge freedom by introducing explicitly $U(1)$-invariant variables.

We note [1] that the direction of the $U(1)$ Cartan subalgebra in the $SU(2)$ Lie algebra can be chosen arbitrarily. However, for simplicity we here proceed with a Cartan subalgebra that coincides with the σ^3 generator of the SU(2) Lie algebra.

Following [1], we introduce the spin and the charge variables

$$X_{\mu+} = \psi_1 e_\mu + \psi_2 e^*_\mu \qquad X_{\mu-} = \psi^*_2 e_\mu + \psi^*_1 e^*_\mu,$$

where the complex field e_μ is a complex combination of zweibeine

$$e_\mu = \frac{e^1_\mu + \mathrm{i}e^2_\mu}{\sqrt{2}} \tag{8}$$

normalized as

$$e_\mu e_\mu = 0 \qquad e_\mu e^*_\mu = 1. \tag{9}$$

[1] We also expect that some other types of decompositions of Yang–Mills variables could have gravitational interpretation [4].

The internal $U(1)_I$ symmetry of the decomposition is defined by the transformations

$$\psi_1 \to \mathrm{e}^{\mathrm{i}\lambda}\psi_1 \tag{10}$$

$$\psi_2 \to \mathrm{e}^{-\mathrm{i}\lambda}\psi_2 \tag{11}$$

$$\phi \to \phi - 2\lambda \tag{12}$$

$$e_\mu \to e_\mu\,\mathrm{e}^{-\mathrm{i}\lambda}. \tag{13}$$

We can factor out the $U(1)_I$ phase variable as follows: if $e_0 \neq 0$, we define η to be the complex phase of e_0 and

$$\hat{e}_\mu = \mathrm{e}^{-\mathrm{i}\eta}e_\mu, \tag{14}$$

if $e_0 = 0$, we can use other charts to define η, so it is a section of $U(1)_I$ bundle over $G(2,4,\mathbb{R}) \times \mathbb{R}^4$. The real Grassmannian $G(2,4,\mathbb{R})$ specifies the 2-plane, in which non-Abelian components of the gauge field lie, for more details see [1].

The variable ρ is defined as

$$\rho^2 = \left(A_\mu^1\right)^2 + \left(A_\mu^2\right)^2 = X_{\mu+}X_{\mu-}. \tag{15}$$

It corresponds to the gauge invariant minimum of the functional (6) prior to any gauge fixing, so it is gauge invariant. We note in passing that it is related to the dimension-2 condensate of [5–8].

An explicit parameterization for $\psi_{1,2}$ in (7) is given by

$$\begin{aligned}\psi_1 &= \rho\,\mathrm{e}^{\mathrm{i}\xi}\cos\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\phi/2}\\ \psi_2 &= \rho\,\mathrm{e}^{\mathrm{i}\xi}\sin\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\phi/2}.\end{aligned} \tag{16}$$

We introduce the $U(1)_I$ gauge field as a Maurer–Cartan 1-form

$$C_\mu = \mathrm{i}\langle e^*, \partial_\mu e\rangle. \tag{17}$$

Define also the $U(1)_I$ invariant gauge field

$$\hat{C}_\mu = \mathrm{i}\langle \hat{e}^*, \partial_\mu \hat{e}\rangle = C_\mu + \partial_\mu\eta \tag{18}$$

and the $U(1) \times U(1)_I$ invariant 3-vector

$$\vec{n} = \begin{pmatrix}\sin\theta\cos(\phi - 2\eta)\\ \sin\theta\sin(\phi - 2\eta)\\ \cos\theta\end{pmatrix} \tag{19}$$

$$n_\pm = n_1 \pm \mathrm{i}n_2 = \sin\theta\,\mathrm{e}^{\pm\mathrm{i}(\phi-2\eta)}. \tag{20}$$

The tensor $P_{\mu\nu}$ is

$$P_{\mu\nu} = \tfrac{1}{2}\rho^2 n_3 \tilde{P}_{\mu\nu}, \tag{21}$$

with $\tilde{P}$ given by

$$\tilde{P}_{\mu\nu} = \mathrm{i}(e_\mu e_\nu^* - e_\nu e_\mu^*). \tag{22}$$

A doublet of $U(1) \times U(1)_I$ invariant, mutually orthogonal and unit normalized 3-vectors p and q can also be used in lieu of the zweibeine:

$$p_i = \tfrac{1}{2}\tilde{P}_{i0}; \qquad q_i = \tfrac{1}{4}\epsilon_{ijk}\tilde{P}_{jk} \tag{23}$$

$$\begin{aligned} &p \cdot q = 0; && p \cdot p + q \cdot q = \tfrac{1}{4} \\ &p \cdot p = \tfrac{1}{2} e_0 e_0^*; && q \cdot q = \tfrac{1}{4} - \tfrac{1}{2} e_0 e_0^*. \end{aligned} \tag{24}$$

Finally, we define the combined $U(1) \times U(1)_I$ covariant derivatives

$$D_\mu \psi_1 = \partial_\mu \psi_1 + \mathrm{i} A_\mu \psi_1 - \mathrm{i} C_\mu \psi_1 \tag{25}$$

$$D_\mu \psi_2 = \partial_\mu \psi_2 + \mathrm{i} A_\mu \psi_2 + \mathrm{i} C_\mu \psi_2 \tag{26}$$

$$D_\mu e_\nu = \partial_\mu e_\nu + \mathrm{i} C_\mu e_\nu \tag{27}$$

and the $U(1) \times U(1)_I$ invariant current

$$J_\mu = \frac{\mathrm{i}}{2\rho^2} \{ \psi_1^* D_\mu \psi_1 - \psi_1 \bar{D}_\mu \psi_1^* + \psi_2^* D_\mu \psi_2 - \psi_2 \bar{D}_\mu \psi_2^* \}. \tag{28}$$

In terms of these variables the standard classical Yang–Mills Lagrangian with a gauge fixing term for the off-diagonal components is [1]

$$L = \frac{1}{4} \left(F_{\mu\nu}^i \right)^2 + \frac{\xi}{2} \left| \nabla_\mu^+ X_{\mu+} \right|^2 \tag{29}$$

$$\begin{aligned} &= \frac{1}{4} \mathcal{F}_{\mu\nu}^2 + \frac{1}{2} (\partial_\mu \rho)^2 + \frac{1}{2} \rho^2 J_\mu^2 + \frac{1}{8} \rho^2 \left(\mathrm{D}_\mu^{\hat{C}} \vec{n} \right)^2 + \rho^2 ((\partial_\mu q)^2 + (\partial_\mu p)^2) \\ &\quad + \frac{1}{4} \rho^2 \left\{ n_+ \left(\partial_a \hat{e}_b \right)^2 + n_- (\partial_a \hat{e}_b)^2 \right\} + \frac{3}{8} \left(1 - n_3^2 \right) \rho^4 - \frac{3}{8} \rho^4 - \frac{1}{2} \partial_\nu \partial_\mu (X_{\mu+} X_{\nu-}), \end{aligned} \tag{30}$$

where in the second equality we have taken the $\xi \to 1$ limit corresponding to the particular MAG condition (4) and

$$\mathcal{F}_{\mu\nu} = \partial_\mu J_\nu - \partial_\nu J_\mu + \frac{1}{2} \vec{n} \cdot \partial_\mu \vec{n} \times \partial_\nu \vec{n} - \{ \partial_\mu (n_3 \hat{C}_\nu) - \partial_\nu (n_3 \hat{C}_\mu) \} - \rho^2 n_3 \tilde{P}_{\mu\nu} \tag{31}$$

$$\left(\mathrm{D}_a^{\hat{C}} \right)^{ij} = \delta^{ij} \partial_a + 2 \epsilon^{ij3} \hat{C}_a \qquad (i, j = 1, 2, 3). \tag{32}$$

3. One instanton configuration in spin–charge separated variables

We recall the explicit BPST one-(anti)instanton solution of the $SU(2)$ gauge theory [9], in the singular gauge it reads

$$A_\mu^a(x) = \varrho^2 \frac{2 \eta_{a\mu\nu} x^\nu}{x^2 (x^2 + \varrho^2)}, \tag{33}$$

where $\eta_{aij} = \epsilon_{aij}$; $\eta_{a4\nu} = -\eta_{a\nu 4} = -\mathrm{i} \delta_{a\nu}$ is the 't Hooft tensor [10] and for simplicity we have centered the instanton at the origin. It is widely known and can be easily verified, that this explicit representation obeys the MAG conditions (4). In particular, the explicit solution (33) is known to be the *global* gauge orbit minimum of the MAG gauge fixing functional

$$\int \mathrm{d}^4 x \rho^2,$$

thus it also obeys the global MAG condition (6). We note that this configuration lies on the Gribov [11] horizon, and the explicit ghost zero modes have been constructed in [12]. Thus (33) is a singular point in the field space, where the fundamental modular region reaches the Gribov horizon.

The instanton solution (33) depends on five parameters, the instanton size ϱ and the four components X_m of the vector that defines the instanton position. These leave the MAG condition intact. Furthermore, when one introduces global $SU(2)$ gauge-rotations in the right-hand side of (33) the total number of free parameters of a single-instanton solution becomes eight. Since a global gauge rotation is equivalent to a spacetime rotation of the solution, the global gauge rotation also does not violate the MAG condition.

Note that, in general, one could expect that the MAG condition fixes the global $SU(2)/U(1)$ gauge freedom. However, for instanton it does not, which also indicates that the instanton is located on the Gribov horizon. We also note that in [13] a continuous family of gauge transformations, smoothly interpolating between regular and singular gauges and satisfying the differential version of the MAG gauge condition has been presented. This suggests that all of them should comprise a valley of degenerate classical solutions for the resulting action. But here we only consider the singular-gauge solution since only this solution lies in the fundamental modular domain.

We now proceed to rewrite the instanton solution in the spin–charge separated variables. We use the relation [10]

$$\eta_{a\mu\nu}\eta_{b\mu\lambda} = \delta_{ab}\delta_{\nu\lambda} + \varepsilon_{abc}\eta_{c\nu\lambda}$$

to get

$$\rho^2 = \frac{8\varrho^4}{(\varrho^2 + x^2)^2 x^2}. \tag{34}$$

The zweibein $e^a_\mu(x)$ defines the 2-plane in which A^a lies

$$e^a_\mu = \eta_{a\mu\nu}\frac{x^\nu}{x}. \tag{35}$$

This choice corresponds to a particular fixing of the $U(1)_I$ gauge freedom: the $U(1)_I$ phase η which is defined in (14), is now constant. Hence it disappears entirely. Note that e^a_μ is ill-defined in the origin but we shall find that the underlying geometry removes this point.

When we choose a gauge where $\psi_{1,2}$ are both real (this fixes both the $U(1)$ and the $U(1)_I$ gauges) we have by definition that

$$A^1_\mu = (\psi_1 + \psi_2)e^1_\mu \tag{36}$$

$$A^2_\mu = (\psi_1 - \psi_2)e^2_\mu. \tag{37}$$

So we find

$$\psi_1 = \rho \qquad \text{and} \qquad \psi_2 = 0 \Rightarrow \theta = 0, \tag{38}$$

which means that the 3-vector $\vec{n}$, defined in (19), points identically to its third (vacuum) direction. Thus this vector field has no classical dynamics.

Generally, $n_3 = \pm 1$ is achieved when $\left(A^1_\mu\right)^2 = \left(A^2_\mu\right)^2$, and a nontrivial $\vec{n}$ accounts for deviations from this equality.

From (23) we calculate the vectors p and q:

$$p_i = \frac{x_\mu x_\nu}{2x^2}(\eta_{1i\nu}\eta_{24\mu} - \eta_{2i\nu}\eta_{14\mu}) \tag{39}$$

$$q_i = \frac{x_\mu x_\nu}{2x^2}\epsilon^{ijk}\eta_{1j\mu}\eta_{2k\nu}. \tag{40}$$

The internal connection (the one characterizing the $U(1)_I$ gauge bundle) defined in (17), is

$$C_\mu = \eta_{3\mu\nu}\frac{x^\nu}{x^2} = \left\{\frac{x_2}{x^2}, -\frac{x_1}{x^2}, \frac{x_4}{x^2}, -\frac{x_3}{x^2}\right\}. \tag{41}$$

The $U(1) \times U(1)_I$ invariant current is simply

$$J_\mu = C_\mu - A^3_\mu. \tag{42}$$

Since $\vec{n}$ is trivial, we conclude that the BPST instanton is also a classical solution to the equations of motion for the *restricted* Yang–Mills Lagrangian

$$L = \tfrac{1}{4}\mathcal{F}^2_{ab} + \tfrac{1}{2}(\partial_a\rho)^2 + \tfrac{1}{2}\rho^2 J^2_a + \rho^2((\partial_\mu p)^2 + (\partial_\mu q)^2) - \tfrac{3}{8}\rho^4 - \tfrac{1}{2}\partial_\nu\partial_\mu(X_{\mu+}X_{\nu-}), \tag{43}$$

where

$$\begin{aligned}\mathcal{F}_{ab} &= \partial_a J_b - \partial_b J_a - \{\partial_a(\hat{C}_b) - \partial_b(\hat{C}_a)\} - \rho^2\tilde{P}_{ab}\\ &= -\partial_a A_b + \partial_b A_a - \rho^2\tilde{P}_{ab}.\end{aligned} \tag{44}$$

Note that for the anti-instanton we find the same solution but with 't Hooft symbol replaced as $\eta \to \bar{\eta}$. To account for arbitrary color orientation of (anti-)instanton, one has to rotate all spacetime indices by the $SU(2)$ matrix acting as $SU(2)_{L(R)}$ subgroup of $SO(4)$ or Lorentz group.

We now turn to the generally covariant formulation of [1]. There, it is proposed that ρ can be viewed as the conformal scale of a conformally flat metric tensor,

$$g = 1_{4\times 4}(\rho/\Delta)^2, \tag{45}$$

where Δ is a constant with the dimension of mass which we need to introduce since ρ is dimension-2 operator while the metric tensor is dimensionless. It is beyond the scope of the present paper to estimate a numerical value for Δ, a possible choice would be to make $\Delta = \Lambda_{QCD}$, alternatively, Δ could coincide with the v.e.v. of dimension-2 condensate (studied in [5–8]), that can be computed. The classical Yang–Mills Lagrangian describes the ensuing locally conformally flat Einstein–Hilbert gravity which is coupled to matter and with a nontrivial cosmological constant. The matter multiplet consists of the massive vector field J_μ, the $O(3)\sigma$-model described by $\vec{n}$ and the nonlinear Grassmannian σ-model $G(4, 2)$ which is described by the zweibeine fields e^a_μ. But in the reduced case (43) the $O(3)$ matter field $\vec{n}$ is decoupled on the classical level (it is classically trivial).

The reduced generally covariant gravitational Lagrangian which is relevant for the present case where $\vec{n}$ is fixed to $\{0, 0, 1\}$ reads [1]

$$\begin{aligned}L = {}&\frac{1}{4}\sqrt{g}g^{\mu\nu}g^{\rho\sigma}\mathcal{F}_{\mu\rho}\mathcal{F}_{\nu\sigma} + \frac{\Delta^2}{12}R\sqrt{g} + \frac{1}{2}\Delta^2\sqrt{g}g^{\mu\nu}J_\mu J_\nu + \Delta^2\cdot\sqrt{g}\cdot g^{\mu\nu}g^{\lambda\eta}\big(\bar{\mathcal{D}}^\sigma_{\mu\lambda}\bar{\mathbb{e}}_\sigma\big)\big(\mathcal{D}^\kappa_{\nu\eta}\mathbb{e}_\kappa\big)\\ &- \frac{3}{8}\Delta^4\sqrt{g} + \frac{1}{2}\partial_\mu(\rho\partial_\mu\rho) - \frac{1}{2}\partial_\nu\partial_\mu(X_{\mu+}X_{\nu-}).\end{aligned} \tag{46}$$

Here $\mathbb{e}_\mu$ is a covariant generalization of the zweibein e^a_μ, obtained with the vierbein E^a_μ:

$$g_{\mu\nu} = \delta_{ab}E^a_\mu E^b_\mu \tag{47}$$

$$\mathbb{e}_\mu = E^a_\mu e_a. \tag{48}$$

Note that the original Yang–Mills indices are here redefined *from Greek to Latin*. All variables are also pulled to the curved space with the help of the vierbein, and we refer to [1] for details and for the full Lagrangian.

Note the appearance of two boundary terms in the action. Both are non-trivial for the instanton solution, and their contribution can be localized to a small sphere that surrounds the singular point $x = 0$ that corresponds to a gauge singularity. The contribution to the action from these two boundary terms cancel each other, they have the exactly opposite limits when $x \to 0$.

We now proceed to the spacetime geometry described by the instanton solution. We find for the metric tensor (45)

$$g = 1_{4\times4}(\rho/\Delta)^2 = H^2\frac{\varrho^4}{(\varrho^2+x^2)^2x^2}1_{4\times4}, \tag{49}$$

where we have introduced a constant $H^2 \equiv 8/(\Delta^2)$.

The scalar curvature of this metric is

$$R(x) = -\frac{18r^4}{H^2\varrho^4} + \frac{36r^2}{H^2\varrho^2} + \frac{6}{H^2}; \qquad r \equiv |x|. \tag{50}$$

In order to inspect the structure of the ensuing manifold, we first rewrite the conformally flat metric in spherical coordinates

$$g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = (\rho/\Delta)^2((\mathrm{d}r)^2 + r^2(\mathrm{d}\Omega_3)^2). \tag{51}$$

After the change of variables

$$r = \frac{\varrho}{\sqrt{\mathrm{e}^{2z}-1}} \tag{52}$$

the metric becomes

$$\left(\frac{\mathrm{d}s}{H}\right)^2 = (\mathrm{d}z)^2 + (1-\,\mathrm{e}^{-2z})^2(\mathrm{d}\Omega_3)^2. \tag{53}$$

When $z \to 0$ which correspond to infinity in the initial coordinates, this metric becomes

$$\left(\frac{\mathrm{d}s}{H}\right)^2 \approx (\mathrm{d}z)^2 + 4z^2(\mathrm{d}\Omega_3)^2. \tag{54}$$

Note the curvature singularity at the point $z \to 0$. In the vicinity of this point the spacetime approaches $\mathbb{R}^4$ in spherical coordinates, but the spherical surfaces $\mathbb{S}^3$ are doubly covered.

When $z \to \infty$, which correspond to the location of the center of the instanton in the original coordinates, the metric rapidly approaches that of $\mathbb{S}^3 \times (0,\infty)$, and the scalar curvature approaches the constant value $6/H^2$. Consequently, asymptotically we have a cylinder of radius H and the geometry is that of an *infinite, asymptotically cylindrical cigar which is doubly wrapped at its beginning*[2]. Note that in terms of the original coordinates the beginning of the cigar corresponds to the spacetime infinity and the infinite cylindrical end of the cigar corresponds to the center of the instanton.

We conclude that the instanton develops a finite radius hole at the position of its center, with the resulting change in the topology and boundary of the spacetime.

Note that the geometry also resolves the directional singularity that we have in the Grassmannian matter field e_μ (see (35)): since the cylindrical cigar has asymptotically constant radius, the Grassmannian vector field becomes well defined. Explicitly, in the new coordinates the covariant version of the Grassmannian zweibein field for $z \gg 1$ reads

$$\mathbb{e} \approx \frac{H\,\mathrm{e}^{\mathrm{i}\phi}}{\sqrt{2}}[\sin\delta(\mathrm{i}\cos\delta + \cos\theta\sin\delta)\sin\theta\,\boldsymbol{d\phi} + (\cos\delta\cos\theta - \mathrm{i}\sin\delta)\sin\delta\,\boldsymbol{d\theta} + \sin\theta\,\boldsymbol{d\delta}], \tag{55}$$

where ϕ, θ, δ are the standard spherical coordinates on $\mathbb{S}^3$. This expression has only complex phase singularities which are irrelevant from the point of view of the Grassmannian σ-model; these singularities only appear in the $U(1)_I$ gauge degree of freedom.

[2] I gratefully acknowledge the help of Joe Minahan in clarifying the questions of geometry.

As $z \to 0$ the zweibein is also non-singular and tends to zero

$$\mathbb{e} \approx Hz\sqrt{2}\,\mathrm{e}^{\mathrm{i}\phi}[\sin\delta(\mathrm{i}\cos\delta + \cos\theta\sin\delta)\sin\theta\,\boldsymbol{d\phi} \\ + (\cos\delta\cos\theta - \mathrm{i}\sin\delta)\sin\delta\,\boldsymbol{d\theta} + \sin\theta\,\boldsymbol{d\delta}]. \tag{56}$$

It is interesting to compare the boundary terms in (46) with the standard boundary term for Einstein gravity for spacetimes with boundaries [14]. There, the boundary terms appear when second derivatives are removed. Here, the action is already in the first-order form and instead we go to the opposite direction,

$$\frac{1}{2}(\partial_\mu\rho)^2 \to \frac{\Delta^2}{12}R\sqrt{g} + \frac{1}{2}\partial_\mu(\rho\partial_\mu\rho). \tag{57}$$

We conclude that the Gibbons–Hawking [14] boundary term is $\frac{1}{2}\partial_\mu(\rho\partial_\mu\rho)$ and *vice versa*,

$$\frac{1}{2}(\rho\partial_\mu\rho) = -\frac{\Delta^2}{6}g_{ab}(\nabla_{e^a}e^b)_\mu = -\frac{\Delta^2}{6}\Gamma^\nu_{\nu\mu}, \tag{58}$$

where e^a are the basis vector fields and Γ is a Christoffel connection. The right-hand side of (58) contracted with the normal vector to the boundary surface is the trace of the second fundamental form and it presents the generally covariant form for the boundary term.

Finally, we only note that the meron solution [15]

$$A^a_\mu(x) = \frac{\bar{\eta}_{a\mu\nu}x^\nu}{x^2} \tag{59}$$

also obeys the MAG condition, and it has the same type of singularity at its center as the instanton. At infinity the geometry is quite different. Since the analysis is straightforward we defer the details.

4. Multi-Instanton case

We now proceed to study the interpretation of a multi-instanton solution. However, due to technical difficulties and transparency of the presentation we only consider explicitly the simplest case, described by an approximate superposition of two instantons. The general case can be investigated by employing the ADHM construction [16] but will not be studied here; we expect that our qualitative results persist in the most general case.

Consider first the non-overlapping limit of two BPST instantons, $y_i \gg \varrho_j$ where y_i are the position 4-vectors for the instantons. In this limit the metric tensor is

$$g = H^2\left(\frac{\varrho_1^2}{(\varrho_1^2 + (x - y_1)^2)|x - y_1|} + \frac{\varrho_2^2}{(\varrho_2^2 + (x - y_2)^2)|x - y_2|}\right)^2 1_{4\times4}. \tag{60}$$

Clearly, near the centers of the instantons the metric is again similar to a cylinder $\mathbb{S}^3 \times (0, \infty)$ of radius H.

We define $\varrho^2 = \varrho_1^2 + \varrho_2^2$ and change the radial variable as $|x| = r = \frac{\varrho}{\sqrt{2z}}$. We also introduce $\mathbb{S}^3$ spherical coordinates for the remaining variables. Note that for the large r the variable z tends to z of the one-instanton case. The metric now becomes

$$\mathrm{d}s^2 = H^2\frac{\varrho^2}{(2z)^3}\left(\frac{\varrho_1^2}{\left(\varrho_1^2 + (x - y_1)^2\right)|x - y_1|} + \frac{\varrho_2^2}{\left(\varrho_2^2 + (x - y_2)^2\right)|x - y_2|}\right)^2 \\ \times ((2z)^2(\mathrm{d}\Omega_3)^2 + (\mathrm{d}z)^2), \tag{61}$$

where $(\mathrm{d}\Omega_3)^2 = (\mathrm{d}\delta)^2 + (\sin\delta\mathrm{d}\theta)^2 + (\sin\delta\sin\theta\,\mathrm{d}\phi)^2$.

To the leading order $r \gg y$ (i.e. $z \ll 1$) this yields

$$\left(\frac{\mathrm{d}s}{H}\right)^2 = ((2z)^2(\mathrm{d}\Omega_3)^2 + (\mathrm{d}z)^2). \tag{62}$$

We here have again the doubly covered sphere $\mathbb{S}^3$ of radius z that characterizes the beginning of the cigar in the single instanton case. As z grows, these spheres become deformed and near each of the instanton centers the metric diverges forming asymptotically an infinite cylinder $\mathbb{S}^3 \times (0, \infty)$ of radius H. Note that the boundary of the space again corresponds to the centers of the instantons, and the singularity in the Grassmannian vector field become resolved in the same manner as in the single instanton case.

5. Monopole loops

It was argued in [13] and also seen directly from various numerical computations [17] that the multi-instanton solutions in the MAG have monopole loop singularities (see also [18] for an analytic example). It means that, really, an instanton ensemble in MAG is not a superposition of singular gauge instantons. Instead, a more general gauge transformation with singularity along some contour (a.k.a. monopole loop) is needed to satisfy the global MAG condition (6). An explicit, analytic form of a monopole loop is not known for a generic instanton ensemble. But for the dilute case it is known that the loops are small, circular and non-percolating (see [13] for explicit expressions).

To construct the monopole loop one applies the gauge transformation

$$\Omega^\dagger = \mathrm{e}^{\mathrm{i}\gamma\tau_3/2}\,\mathrm{e}^{\mathrm{i}\beta\tau_2/2}\,\mathrm{e}^{\mathrm{i}\alpha\tau_3/2},$$

where

$$\alpha = \phi - \psi\gamma = \pi - \phi - \psi \tag{63}$$

and

$$\beta = 2\theta - \arctan\frac{u}{v+R} - \arctan\frac{u}{v-R} \tag{64}$$

to the anti-instanton in the singular gauge (or its conjugate Ω to the instanton). Here the coordinates

$$x_1 + \mathrm{i}x_2 = u\mathrm{e}^{\mathrm{i}\phi} \qquad x_3 + \mathrm{i}x_4 = v\mathrm{e}^{\mathrm{i}\psi}$$

are used. This gives a monopole loop of radius R that lies in the (3–4) plane and can be arbitrarily reoriented by a global gauge transformation.

The corresponding value of ρ^2 is

$$\rho^2 = \frac{8((\varrho^2 - R^2)^2u^2 + (\varrho^2 + R^2)^2v^2)}{(\varrho^2 + u^2 + v^2)^2(R^4 + 2(u^2 - v^2)R^2 + (u^2 + v^2)^2)}. \tag{66}$$

At $R = 0$ the above equation reduces to (34) for the singular gauge instanton. The singularity here forms a ring of radius R. Near this ring the condensate is

$$\rho^2 = \frac{2}{z^2} + \frac{2(\varrho^2 - 3R^2)\cos(\eta)}{R(\varrho^2 + R^2)z} + \mathcal{O}(z^0), \tag{67}$$

where we used natural coordinates for a ring: $z\cos\eta = (v - R)$; $z\sin\eta = u$ with $\eta \in [0, \pi)$.

The 3-vector n is again trivial $n = \{0, 0, 1\}$, so

$$\psi_1 = \rho \qquad \psi_2 = 0$$

and the zweibein field is simply related to the new gauge transformed potential

$$e_\mu = \frac{A^1_\mu + \mathrm{i}A^2_\mu}{\sqrt{2}\rho}. \tag{68}$$

Keeping only the first term in (67) we analyze the qualitative behavior of the corresponding geometry. In coordinates ψ (angular position on the ring), z, η, ϕ (spherical coordinates around ring points) the metric associated with ρ^2 becomes

$$\mathrm{d}s^2 \approx \frac{2}{\Delta^2 z^2}(R^2\,\mathrm{d}\psi^2 + \mathrm{d}z^2 + z^2\,\mathrm{d}\eta^2 + z^2\sin^2\eta\mathrm{d}\phi^2). \tag{69}$$

Changing the variable $z = H\mathrm{e}^{-y}$ results in

$$\left(\frac{\mathrm{d}s}{H}\right)^2 \approx \frac{1}{4}(\mathrm{e}^{2y}(R/H)^2\,\mathrm{d}\psi^2 + \mathrm{d}y^2 + \mathrm{d}\eta^2 + \sin^2\eta\,\mathrm{d}\phi^2). \tag{70}$$

Thus the space is like infinite cylinders $(\eta, \phi, y) \in \mathbb{S}^2 \times (0, \infty)$ of radius $H/2$ over each point of the circle, parameterized by ψ. Cylinders over different points spread away from each other exponentially fast as one moves along them (i.e. increases y). The scalar curvature tends to zero exponentially fast with increasing of y.

When we are far from the monopole loop, the results obtained for the singular gauge are valid, thus the total geometry looks like $\mathbb{S}^3$-cylinder that becomes split into infinite $\mathbb{S}^2$ cylinders. In terms of the variable z introduced for analysis of singular gauge instanton in (52), the splitting occurs at $z \approx \log(\varrho/R)$. The resulting cylinders with $\mathbb{S}^2$ profiles are fibered over one of the $\mathbb{S}^1$ sections of $\mathbb{S}^3$, depending on the orientation of the monopole loop. This splitting can be illustrated in lower dimensions in terms of gradual transition from a sphere to a torus.

In the coordinates $\{y, \psi, \eta, \phi\}$, used above, the Grassmannian field $\mathbb{e}$, defined in (48), takes the form

$$\mathbb{e} = \frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}(\phi+\psi)}H(\varrho^2 - R^2)\sin(\eta)}{2(\varrho^2+R^2)}\mathbf{d}\psi + \frac{\mathrm{e}^{\mathrm{i}(\phi+\psi)}H}{2}\mathbf{d}\eta + \frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}(\phi+\psi)}H\sin(\eta)}{2}\mathbf{d}\phi + \mathcal{O}(\mathrm{e}^{-y}). \tag{71}$$

We again observe that the geometry resolves a directional singularity of the initial field e_μ: the new field $\mathbb{e}$, defined on the space with metric (45), is non-singular. The singularity near the monopole loop is mapped to a boundary surface at infinity.

6. Discussion

In summary, we have elaborated on the observation [1] that the Yang–Mills action in the MAG gauge can be viewed as a locally conformally flat limit of a gravitational theory that involves the Einstein action and a cosmological constant in interaction with matter fields.

We have inspected the BPST instanton in this formalism and found that the spacetime has the shape of a doubly wrapped cigar. For a multi-instanton, we expect a spacetime with several cigar-like cylinders, growing from different singular points as deformations of the space. These cylinders asymptotically approach $(0, a) \times \mathbb{S}^3$ and then split into $(a, \infty) \times \mathbb{S}^2$ cylinders fibered over $\mathbb{S}^1 \subset \mathbb{S}^3$.

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